

# FUNDAMENTAL FUNCTION FOR GRAND LEBESGUE SPACES.

E.Ostrovsky, L.Sirota

Department of Mathematic, Bar-Ilan University, Ramat Gan, 52900, Israel,  
e-mails: eugostrovsky@list.ru, sirota3@bezeqint.net

ABSTRACT.

We investigate in this short article the fundamental function for the so-called Grand Lebesgue Spaces (GLS) and show in particular a one-to-one and mutually continuous accordance between its fundamental and generating function.

*Key words and phrases:* Young-Orlicz function, ordinary and Grand Lebesgue Spaces (GLS); Orlicz, GLS norms, rearrangement invariant spaces, fundamental and generating function, Young-Fenchel, or Legendre transform, theorem of Fenchel-Morau, inverse function, Exponential Orlicz function (EOF) and Spaces (EOS).

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## 1 Notations. Statement of problem.

**A.** A triplet  $(X, \mathcal{B}, \mu)$ , where  $X = \{x\}$  is arbitrary set,  $\mathcal{B}$  is non-trivial certain sigma-algebra of subsets  $X$  and  $\mu$  is probabilistic:  $\mu(X) = 1$  diffuse non-negative completely additive measure defined on the  $\mathcal{B}$ .

The non-probabilistic case  $\mu(X) = \infty$  will be consider further.

Recall that the measure  $\mu$  is said to be diffuse iff for arbitrary measurable set  $A_1 \in \mathcal{B}$  with positive measure:  $\mu(A_1) > 0$  there exists it subset  $A_2 \subset A_1$  such that  $\mu(A_2) = \mu(A_1)/2$ .

We denote as usually for any arbitrary measurable function  $f : X \rightarrow R$

$$|f|_p = \left[ \int_X |f(x)|^p \mu(dx) \right]^{1/p}, \quad p \geq 1;$$

$$L_p = \{f, |f|_p < \infty\}.$$

**B.** The so-called Grand Lebesgue Space (GLS)  $G\psi$  with norm  $\|\cdot\|_{G\psi}$  is defined ( not only in this article) as follows:

$$G\psi = \{f, \|f\|_{G\psi} < \infty\}, \quad \|f\|_{G\psi} \stackrel{def}{=} \sup_{p \geq 1} \left[ \frac{|f|_p}{\psi(p)} \right]. \quad (1.1)$$

Here  $\psi = \psi(p)$ ,  $1 \leq p < \infty$  is some continuous strictly increasing function such that  $\lim_{p \rightarrow \infty} \psi(p) = \infty$ .

The detail investigation of this spaces (and more general spaces) see in [14], [19]. See also [5], [6], [8], [9], [10] etc.

The case when in (1.1) supremum is calculated over *finite* interval is investigated in [14], [19], [20]:

$$G_b\psi = \{f, \|f\|_{G_b\psi} < \infty\}, \quad \|f\|_{G_b\psi} \stackrel{\text{def}}{=} \sup_{1 \leq p < b} \left[ \frac{|f|_p}{\psi(p)} \right], \quad b = \text{const} > 1, \quad (1.2)$$

but in (1.2)  $\psi = \psi(p)$  is continuous function in the semi-open interval  $1 \leq p < b$  such that  $\inf_{p \in (1, b)} \psi(p) > 0$ .

We will denote

$$[1, b) := \text{supp } \psi(\cdot),$$

or simple  $b = b(\psi) := \text{supp } \psi(\cdot)$ , including the case  $b = \infty$ .

**Definition 1.1.** The function  $\psi(p)$  which appeared in (1.1) and (1.2), will be named as *generating function* for the correspondent Banach space  $G\psi$ .

An used further example:

$$\psi^{(\beta, b)}(p) = (b - p)^{-\beta}, \quad 1 \leq p < b, \beta = \text{const} > 0; \quad b = \text{const} > 1,$$

$$G_{\beta, b}(p) := G_b\psi^{(\beta, b)}(p).$$

**C.** We denote as ordinary for any measurable set  $A$ ,  $A \in \mathcal{B}$  its indicator function by  $I(A) = I_A(\omega)$ .

**D.** The Grand Lebesgue Spaces  $\{G\psi\}$  are rearrangement invariant in the classical definition, see e.g. [2], chapter 1. Therefore, its fundamental function  $\phi_{G(\psi)}(\delta)$ ,  $\delta \geq 0$  is correctly defined in the considered case as follows:

$$\phi_{G(\psi)}(\delta) \stackrel{\text{def}}{=} \sup_{p \in \text{supp } \psi} \left[ \frac{\delta^{1/p}}{\psi(p)} \right], \quad (1.3)$$

see [2], chapters 2 and 5.

For instance,

$$\phi_{G(\psi)}(1) = \frac{1}{\inf_{p \in \text{supp } \psi} \psi(p)}. \quad (1.3a)$$

Note also

$$\phi_{G(C \cdot \psi)} = \phi_{G(\psi)} / C, \quad C = \text{const} > 0. \quad (1.3b)$$

This notion play a very important role in the functional analysis, [2], [22], [23]; in the theory of interpolation of operators, [2], [5], [7], in the theory of probability [13], [15], [16], [17]; in the theory of Partial Differential equations [7], [9]; in the

theory of martingales [20]; in the theory of approximation, in the theory of random processes etc.

**E.** Let  $g = g(p)$ ,  $p \in (a, b)$ ,  $1 \leq a < b \leq \infty$  be some numerical valued continuous strictly increasing (or decreasing) function. The *inverse* function will be denoted by  $g^{(-1)}(z)$ ,  $g(a) \leq z \leq g(b)$ , in contradistinction to the usually notation  $g^{-1}(p) = 1/g(p)$ .

**F.** The Young-Fenchel, or Legendre transform  $g^*(q)$  for the function  $g = g(p)$  one can to define

$$g^*(q) \stackrel{def}{=} \sup_{p \in \text{supp } g} (p|q| - g(p)). \quad (1.4)$$

**Our goal in this short report is to establish a one-to-one and mutually continuous connection between the fundamental and generating functions for the Grand Lebesgue Spaces.**

In some previous works: [6], [12], chapter 8; [14], [19], [18], [22] these function was evaluated and applied in many practical cases.

## 2 Main result.

**Problem A.** Let the generating function  $\psi$  be a given:  $\psi \in G\Psi(a, b)$ ,  $1 \leq a < b \leq \infty$ . Find the fundamental function for the correspondent Grand Lebesgue Space  $G\psi$ .

Suppose the function

$$p \rightarrow \frac{p}{\psi(p)}, \quad p \in (a, b)$$

is strictly increasing; and define therefore the function

$$\nu(p) = \nu_\psi(p) = \left[ \frac{p}{\psi(p)} \right]^{(-1)}, \quad p \in \text{supp } \psi, \quad (2.1)$$

and  $\nu(p) = +\infty$  otherwise.

Introduce also the following Young-Orlicz function

$$N(u) = N_\psi(u) := \exp \left( \nu_\psi^*(u) \right) - \exp \left( \nu_\psi^*(0) \right), \quad (2.2)$$

and define finally

$$\theta(\delta) = \theta_\psi(\delta) \stackrel{def}{=} \frac{1}{N_\psi^{(-1)}(1/\delta)}, \quad \delta > 0. \quad (2.3)$$

**Proposition 2.1.** *We propose under formulated above conditions, for instance,  $\mu(X) = 1$ , diffuseness of the measure  $\mu$ , and in the case when  $b = \infty$*

$$\phi_{G(\psi)}(\delta) = \theta_\psi(\delta), \quad \delta > 0. \quad (2.4)$$

**Remark 2.1.** The equality (2.4) is more convenient than source definition (1.3). In particular, it allows for a relatively simple inversion.

**Proof** is very simple; it based on the computation of the fundamental function for Orlicz spaces, see the book of Krasnosel'skii M.A. and Rutickii Ya.B. [11], chapter 3; see also the classical monographs [26], [27].

In detail, it is proved in particular in the articles [14], [18], [19] that under our conditions the Grand Lebesgue Space  $G\psi$  coincides with certain Orlicz space over source probability triplet  $(X, \mathcal{B}, \mu)$  relative the Young-Orlicz function  $N_\psi(u)$ .

We deduce reducing considered case to the well-known calculation of fundamental function for Orlicz space, [11], chapter 3

$$\phi_{G(\psi)}(\delta) = \frac{1}{N_\psi^{(-1)}(1/\delta)} = \theta_\psi(\delta), \quad \delta > 0, \quad (2.4)$$

Q.E.D.

**An inverse problem B.** Let the fundamental function  $\phi_{G\psi}(\delta) = \phi(\delta)$  be a given. Find the correspondent generating function  $\psi(p)$ .

A first restrictions: the function  $\phi = \phi(\delta)$  is strictly increasing and continuous; in particular  $\phi(0+) = \phi(0) = 0$ .

We find from the equality (2.4)

$$N_\psi(1/\delta) = \left( \frac{1}{\phi(\delta)} \right)^{(-1)}, \quad (2.5)$$

or equivalently

$$N_\psi(z) = \left( \frac{1}{\phi(\delta)} \right)^{(-1)} \Big|_{\delta:=1/z}. \quad (2.5a)$$

A second restriction: the function

$$V^*(z) = \ln(C + N_\psi(z)), \quad z \geq 0, \quad (2.6)$$

where  $V^*(0) = \ln C$ , is continuous and upward convex.

It follows immediately from (2.6) by virtue of theorem of Fenchel-Morau

$$V(z) = \{\ln(C + N_\psi(z))\}^*, \quad z \geq 0. \quad (2.7)$$

Since

$$V(p) = \left[ \frac{p}{\psi(p)} \right]^{(-1)},$$

we derive finally

**Proposition 2.2.** We conclude under formulated in this pilcrow conditions

$$\psi(p) = \frac{p}{V^{(-1)}(p)}. \quad (2.8)$$

### 3 The case of infinite measure.

The case when  $\mu(X) = \infty$  is more complicated.

Recall first of all definition and some facts about the so-called Exponential Orlicz Spaces (EOS), see for example [21].

Let  $N = N(u)$  be an  $N$  – Young-Orlicz's function, i.e. downward convex, even, continuous function differentiable for all sufficiently greatest values  $u$ ,  $u \geq u_0$ ,  $u_0 = \text{const} > 0$ , strongly increasing along the right semi-axis and such that  $N(u) = 0 \Leftrightarrow u = 0$ ;  $u \rightarrow \infty \Rightarrow dN(u)/du \rightarrow \infty$ . We can say that  $N(\cdot)$  is an exponential Orlicz function, briefly,  $N(\cdot) \in EOF$ , if  $N(u)$  has a form of a continuous differentiable strongly increasing downward convex function  $W = W(u)$  in the domain  $[2, \infty]$  such that  $u \rightarrow \infty \Rightarrow W'(u) \rightarrow \infty$  and

$$N(u) = N(W, u) = \exp(W(\log |u|)), \quad |u| \geq e^2.$$

For the values  $u \in [-e^2, e^2]$  we define  $N(W, u)$  arbitrarily, but so that the function  $N(W, u)$  is even, continuous, convex, strictly increasing along the right semi-axis and so that  $N(u) = 0 \Leftrightarrow u = 0$ . We denote the correspondent Orlicz space on  $(X, \mu)$  with a measure  $\mu$  and with  $N$  – function of the form  $N(W, u)$  as  $L(N) = EOS(W)$ ;  $EOS = \cup_W \{EOS(W)\}$  (exponential Orlicz's space).

For example, let  $m = \text{const} > 0$ ,  $r = \text{const} \in \mathbb{R}^1$ ,

$$N_{m,r}(u) = \exp \left[ |u|^m \left( \log^{-mr}(C_1(r) + |u|) \right) \right] - 1,$$

$C_1(r) = e$ ,  $r \leq 0$ ;  $C_1(r) = \exp(r)$ ,  $r > 0$ . Then  $N_{m,r}(\cdot) \in EOS$ . In the case  $r = 0$  we can write  $N_m = N_{m,0}$ .

Recall that the Orlicz's norm on the arbitrarily measurable space  $(X, A, \mu)$   $\|f\|L(N) = \|f\|L(N, X, \mu)$  can be calculated by the following formula (see, for example, [11], p. 66; [26], p. 73 )

$$\|f\|L(N) = \inf_{v>0} \left\{ v^{-1} \left( 1 + \int_X N(v|f(x)|) \mu(dx) \right) \right\}.$$

Let  $\alpha$  be arbitrary number,  $\alpha = \text{const} \geq 1$ , and  $N(\cdot) \in EOS(W)$  for some  $W = W(\cdot)$ . For such a function  $N = N(W, u)$  we denote by  $N^{(\alpha)}(W; u) = N^{(\alpha)}(u)$  a new Young-Orlicz's function  $N^{(\alpha)}(u)$  such that

$$\begin{aligned} N^{(\alpha)}(u) &= C_1 |u|^\alpha, \quad |u| \in [0, C_2]; \\ N^{(\alpha)}(u) &= C_3 + C_4 |u|, \quad |u| \in (C_2, C_5]; \\ N^{(\alpha)}(u) &= N(u), \quad |u| > C_5, \quad 0 < C_2 < C_5 < \infty, \\ C_{1,2,3,4,5} &= C_{1,2,3,4,5}(\alpha, N(\cdot)). \end{aligned} \quad (3.1)$$

In the case of  $\alpha = m(j+1)$ ,  $m > 0$ ,  $j = 0, 1, 2, \dots$  the function  $N_m^{(\alpha)}(u)$  is equivalent to the following Trudinger's function:

$$N_m^{(\alpha)}(u) \sim N_{[m]}^{(\alpha)}(u) = \exp(|u|^m) - \sum_{l=0}^j u^{ml}/l!.$$

This method is described in [28], p. 42-47. These Orlicz spaces are applicable to the theory of non-linear partial differential equations.

We denote hereinafter generally by  $C_k = C_k(\cdot)$ ,  $k = 1, 2, \dots$  some positive finite essentially constructive constants, and by  $C, C_0$  non-essentially constants, also constructive. We proved the existence of constants  $C_{1,2,3,4,5} = C_{1,2,3,4,5}(\alpha, N(\cdot))$  such that  $N^{(\alpha)}$  is a new exponential  $N$  Orlicz's function in [21]. We denote classical absolute constants by the symbols  $K_j$ .

Now we introduce some *new* Grand Lebesgue Spaces. Let  $\psi = \psi(p)$ ,  $p \geq \alpha$ ,  $\alpha = \text{const} \geq 1$  be a continuous positive  $\psi(\alpha) > 0$  finite strictly increasing function such that the function  $p \rightarrow p \log \psi(p)$  is downward convex, and

$$\lim_{p \rightarrow \infty} \psi(p) = \infty.$$

We denote the set of all these functions by  $\Psi$ ;  $\Psi = \{\psi\}$ . A particular case

$$\psi(p) = \psi(W; p) = \exp(W^*(p)/p),$$

where

$$W^*(p) = \sup_{z \geq \alpha} (pz - W(z))$$

is so-called Young-Fenchel, or Legendre transform of  $W(\cdot)$ . It follows from the theorem of Fenchel-Morauux that in this case

$$W(p) = [p \log \psi(W; p)]^*, \quad p \geq p_0 = \text{const} \geq 2,$$

and, consequently, for all  $\psi(\cdot) \in \Psi$  we introduce a correspondent Young-Orlicz  $N$  – function by the equality:

$$N([\psi]) = N([\psi], u) = \exp \{ [p \log \psi(p)]^* (\log u) \}, \quad u \geq e^2.$$

**Definition 3.1.** We introduce for such arbitrary function  $\psi(\cdot) \in \Psi$  the so-called  $G(\alpha; \psi)$  norms and correspondent Banach GLS space  $G(\alpha; \psi)$  as a set of all measurable (complex) functions with finite norms:

$$\|f\|_{G(\alpha; \psi)} = \sup_{p \geq \alpha} (|f|_p / \psi(p)). \quad (3.2)$$

For instance,  $\psi(p)$  may be  $\psi(p) = \psi_m(p) = p^{1/m}$ ,  $m = \text{const} > 0$ ; in this case, we can write  $G(\alpha, \psi_m) = G(\alpha, m)$  and

$$\|f\|_{G(\alpha, m)} = \sup_{p \geq \alpha} (|f|_p p^{-1/m}).$$

**Theorem A**, see [21]. *Let the measure  $\mu$  be diffuse,  $\mu(X) = \infty$ ,  $\alpha = \text{const} \geq 1$  and  $\psi \in \Psi$ . We assert that the norms Orlicz-Luxemburg norm  $\|\cdot\|_{L(N^{(\alpha)}, [\psi])}$  and Grand Lebesgue Space norm  $\|\cdot\|_{G(\alpha, \psi)}$ ,  $\alpha \geq 1$  are equivalent.*

Arguing similarly to the second section, we obtain the following result.

**Proposition 3.1.** *We propose under conditions of theorem A*

$$\phi_{G(\psi)}(\delta) \asymp \theta_{\psi}(\delta), \quad \delta > 0.$$

**Remark 3.1.** Note that this case  $\delta \in (0, \infty)$ , in contradiction to the proposition 2.1, where it is naturally to take  $\delta \in (0, 1)$ .

## 4 Concluding remarks. Open problems.

It is interesting by our opinion to investigate the notion of fundamental function and also its relation with generating function for the so-called mixed, or equally anisotropic Grand Lebesgue Spaces.

Recall that the definition of mixed, or equivalently anisotropic ordinary Lebesgue Spaces appeared at first in the article [1] and was investigated in detail in the classical books [3], [4].

The anisotropic Grand Lebesgue Spaces as a slight generalization of  $L_p$  spaces arises in turn in [23] with the correspondent fundamental function; in the preprint [24] both this notions was applied in the operator's theory.

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